Gravitational instanton, inflation and cosmological constant

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Quantum fluctuation of unstable modes about gravitational instantons causes the instability of flat space at finite temperature, leading to the spontaneous process of nucleating quantum black holes. The density of vacuum energy-gain in such process gives the cosmological term in the Einstein equation. This naturally results in the inflationary phase of Early Universe. While the reheating phase is attributed to the Hawking radiation of these quantum black holes. In the Standard cosmology era, this cosmological term depends on the reheating temperature and asymptotically approaches to the cosmological constant in matter domination phase, consistently with current observations.

PACS numbers: 98.80.H, 03.70.+k, 04.70.Dy

Introduction. In recent years, as observational data concerning on cosmology are rapidly accurated, the theoretical understanding of our Universe has been greatly profounded ever before. The inflation[1] in early Universe and the acceleration of present Universe[2] are two most important issues in modern cosmology and fundamental physics. Both issues are closely related to the vacuum-energy represented by the cosmological constant in Einstein equation. Much effort in understanding these issues has been made for many decades and there are many interesting ideas and innovative theoretical developments based on either simple models[3] or complex theories[4]. In these approaches, a scalar field slowly-rolling downwards an effective potential, which mimics a possible vacuum-energy variation, plays a crucial rôle in driving inflation; whereas this scalar field with a mass of order the current Hubble scale, can possibly account for acceleration. We attempt to study these two issues within a framework based on the vacuum-energy variation in spacetime decay due to an unstable quantum fluctuation about gravitational instantons.

Instability of flat space at finite temperature. The attractive nature of gravity that cannot be screened is the essential reason for many inevitable instabilities of classical gravitating systems. One might worry about flat-space metastable state and the stability of flat space against quantum-field tunneling. The positive-energy theorem *á la* Schoen and Yau[5] shows that the total energy of asymptotically flat manifolds is positively semidefinite and only Minkowski space has zero energy.

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This precludes the possibility of flat space at zero temperature decaying by any mechanism.

In Ref.[6], authors explored two instabilities of flat space at finite temperature T, which is attributed to a thermal equilibrium of gravitons and other particles. For large-wavelength density fluctuations of the thermal gravitons, a Jeans instability is expected to occur. Another source of instability is the nucleation of quantum black holes, due to an effect of quantum fluctuations about a gravitational instanton, which is the the Euclidean section of the Schwarzschild metric g^S_{ab}. The classical Euclidean action of N non-interacting instantons is given by,

$$I^n_i = NI_i, \quad I_i = \frac{1}{2}\beta M, \quad \beta = (8\pi GM), \tag{1}$$

where I_i is the classical Euclidean action for an instanton, M is not a fixed mass, but rather determined in terms of the temperature $T = \beta^{-1}$. The inverse temperature β is the period in Euclidean time.

About this instanton saddle point g^S_{ab}, in addition to quantum fluctuations of stable modes of thermal gravitons, there is an unstable (negative) mode $(\epsilon\tilde{\phi}_{ab})$[6, 7], namely $g_{ab} = g^S_{ab} + \epsilon\tilde{\phi}_{ab}$. Its quantum fluctuation decreases the classical action (1) by $\delta I_i = -9.4 \cdot 10^{-4}\epsilon^2/(GM)^2$. Its contributions to the partition functional integral give rise to an imaginary part in the free energy (effective action). In Ref.[6], this is interpreted as a finite lifetime for decay of flat space at finite temperature T. The decay proceeds by quantum fluctuations spontaneously nucleating quantum black holes of radius $R = (4\pi T)^{-1}$ and mass $M = (8\pi GT)^{-1}$, where the Planck mass $m_p = G^{-1/2}$. Assuming the configuration of non-interacting instantons, one calculated the rate (per unit volume δV) for nucleating these quantum black holes[6, 8]

$$\Gamma(T) \equiv \frac{\delta^2 N}{\delta t \delta V} = 0.87T(\frac{m_p}{T})^\theta \frac{m_p^3}{64\pi^3} \exp\Big[-\frac{m_p^2}{16\pi T^2}\Big], \tag{2}$$

$$\theta = \frac{1}{45}(212n_2 - \frac{233}{4}n_{3/2} - 13n_1 + \frac{7}{4}n_{1/2} + n_0), \tag{3}$$

where n_s is the number of massless spin-s fields. At high-temperature $T \sim m_p$, the rate (2) of the nucleation is maximum and exponentially suppressed at low-temperature $T \ll m_p$.

This shows that (i) stable and unstable quantum fluctuations of gravitational field contribute to the vacuum energy and (ii) the vacuum of flat space at finite temperature T is energetically unstable and decays by spontaneously producing quantum black holes. The energy-gain in this vacuum-decay should have an important impact on the evolution of early Universe and be related to the cosmological constant of present Universe.

Black hole nucleation and cosmological term. Based on Eq.(2), the total number N of quantum

black holes nucleated is given by

$$N = \int (-g)^{1/2} d^4x \Gamma(T). \tag{4}$$

We assume that (i) these quantum black holes are free from interacting each other and their spatial distribution is homogeneous; (ii) the kinetic energy of these quantum black holes is much smaller than their mass-energy M. Analogously to Eq.(1), the effective Euclidean action of these non-interacting black holes is given by using Eq.(4),

$$I_{\rm BH} = \int (-g)^{1/2} d^4x \Gamma(T) \frac{1}{2}\beta M. \tag{5}$$

The corresponding effective action in Minkowski time t is obtained by the Wick rotation $\beta = \int d\beta = -i \int dt$,

$$S_{\rm BH} = \int (-g)^{1/2} d^4x \rho_\Lambda(t), \tag{6}$$

where

$$\begin{aligned} \rho_\Lambda(t) &= \frac{1}{2}\int_{t_0}^{t} d\tau M(\tau)\Gamma[T(\tau)], \\ &= \frac{0.87 m_p^4}{1024\pi^4}\int_{t_0}^{t} d\tau \left(\frac{1}{T(\tau)}\right)^{\theta} e^{-\frac{1}{16\pi T^2(\tau)}} \end{aligned} \tag{7}$$

which is the energy-density of quantum black holes nucleated from initial time t_0 and temperature T_0 to final time t and temperature $T(t)$. The energy-gain $\delta E_\Lambda = \rho_\Lambda \delta V$ in the spontaneous nucleation process. In Eq.(7) and henceforth, the temperature T and time t are in Planck units.

Armed with the effective action (6) together with actions for gravity and thermal particles, we arrive at the Einstein equation

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = 8\pi G T_{ab}, \tag{8}$$

where the cosmological term $\Lambda = 8\pi G \rho_\Lambda$ is originated from the effective action (6). Such cosmological term has two features: (i) its geometric origin from gravitational instantons, appearing in the l.h.s. of Eq.(8); (ii) its energetic origin from the energy-gain δE_Λ of nucleation of quantum black holes $\delta E_\Lambda = -\delta E$, where $\delta E < 0$ is the vacuum-energy variation of the flat space at finite temperature. The second point gives negative pressure $p_\Lambda \equiv -\delta E_\Lambda/\delta V = -\rho_\Lambda$. The corresponding energy-momentum tensor $T^\Lambda_{ab} = -g_{ab}\rho_\Lambda$. The total energy conservation $(T_{\mu\nu} + T^\Lambda_{\mu\nu})^{;\mu} = 0$.

Inflation in early Universe. We consider the post-Planckian era of early Universe, for the time $t \geq t_0 = 1$. The flat space is at the temperature $T \leq T_0 = 1$. The spontaneously energy-gain

process of nucleating quantum black holes is bound to occur. Suppose that the Universe is given by the Robertson-Walker line element with the scale factor $a(t)$ and zero curvature $k = 0$, then the Einstein equation (8) describing Universe expansion becomes,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m(t) + \rho_\Lambda(t)). \tag{9}$$

where ρ_m is the energy-densities of thermal particles and $\rho_\Lambda(t)$ is given by Eq.(7) with the initial time $t_0 = 1$ and temperature $T_0 = 1$. The Universe expands and its temperature $T(t)$ decreases, $a(t)T(t)$ =constant for the entropy-conservation. As results, the energy-densities $\rho_m(t)$ decreases, whereas $\rho_\Lambda(t)$ (7) asymptotically approaches a constant $\bar{\rho}_\Lambda$ for $t \gg 1$. This implies an inflationary Universe. How does such inflation end?

Hawking radiation and reheating. When the Universe temperature $T(t)$ is smaller than the temperature $T(\tau) = 1/(8\pi GM(\tau))$ of quantum black holes that are created at an earlier time $\tau < t$, these quantum black holes loss their masses by the Hawking radiation. On the other hand, accretion occurs if $T(t) > T(\tau)$. If we were only to consider emission and absorption of gravitons, the mass-variation of quantum black holes is given by[9, 10]

$$\frac{\delta M(\tau)}{\delta\tau} = \frac{\pi^2}{15}[T^4(t) - T^4(\tau)]4\pi R^2(\tau), \tag{10}$$

where the black hole size $R(\tau) = 2M(\tau)$. For $T(\tau) \gg T(t)$, we approximately obtain,

$$M_H(t) \simeq M(\tau)[1 - \frac{2\pi^2}{5}T^3(\tau)(t-\tau)]^{1/3}, \tag{11}$$

indicating that quantum black holes loss mass. We speculate that the extremal minimum of black hole mass is the order of the Planck scale, where Eqs.(10,11) are not applicable.

The Hawking process (11) occurs, contemporaneously with quantum black hole nucleation. Due to the Hawking radiation, the energy-density $\rho_\Lambda(t)$ (7) of quantum black holes is reduced by

$$\begin{aligned} \Delta\rho_\Lambda(t) &= \frac{1}{2}\int_{t_0}^t d\tau \Delta M \Gamma[T(\tau)], \\ &= \frac{0.87 m_p^4}{128\pi^3}\int_{t_0}^t d\tau \Delta M \left(\frac{1}{T(\tau)}\right)^{(\theta-1)} e^{-\frac{1}{16\pi T^2(\tau)}}, \end{aligned} \tag{12}$$

where $\Delta M = M_H(t) - M(\tau)$ and $M_H(t)$ is given by Eq.(11). The energy-density $\rho_\Lambda(t)$ in Eq.(9) should be replaced by

$$\rho(t) = \rho_\Lambda(t) + \Delta\rho_\Lambda(t) < \rho_\Lambda(t). \tag{13}$$

This shows that the energy-gain $\rho_\Lambda(t)$ in quantum black hole nucleation is converted into the radiation energy $\Delta\rho_\Lambda(t)$, which reheats the Universe.

Assuming $a(t)T(t) \simeq 1$ and $\theta = 212/45$(graviton), $t_0 \simeq 1$ and $T_0 \simeq 1$, we numerically integrate Eqs.(9,13) for the evolution of Universe, by taking into account both the nucleation process (2,7) and the Hawking process (11,12). At the beginning ($t < O(10^2)$), the nucleation process is dominate over the Hawking process. The variation of energy-density $\rho(t)$ (13) is rather slow, so that the "graceful exit" problem is avoided and the evolution of Universe is inflationary. At the end ($t > O(10^2)$), however, the Hawking process eventually ends the inflation and reheats Universe.

Numerical results. In the initial phase $t \sim 1$, the energy-density of quantum black holes $\rho_\Lambda(t)$ (7) is negligible, compared with the energy-density of thermal graviton gas $\rho_m = \frac{\pi^2}{15}T^4$. The solution to Eq.(9) is radiative,

$$a_1 = a_0 t^{1/2}, \quad T_1 = T_0 t^{-1/2}, \tag{14}$$

where $a_0 \sim 1$ and $T_0 \sim 1$ are initial scaling-factor and temperature. $a_0 T_0 \sim 1$ implies that the initial entropy $S_0 = (a_0 T_0)^3$ is given by $O(1)$ quantum states of Planck energy in the Planck volume. As the time t increases, $\rho_m(t)$ decreases, $\rho_\Lambda(t)$ increases and becomes dominant in Eq.(9).

In Fig.[1], we plot $\rho_m(t)$ (cross line) and $\rho_\Lambda(t)$ (short-dash line) in terms of the time t. It is shown that (i) the pre-inflationary phase (14) for $t < 10$; (ii) the inflationary phase for $t > 10$, where ρ_m is vanishing and $\rho_\Lambda(t)$ is approaching to an asymptotic value $\bar{\rho}_\Lambda \simeq 9.03 \cdot 10^{-2} m_p^4$. In Fig.[1], we also plot the radiation energy $|\Delta\rho_\Lambda(t)|$ (12) (dot-line) and the energy-density $\rho(t)$ (13) (long-dash line), showing that $\rho(t)$ slowly varies for $8 < t < 100$ and $\rho(t) \to 0$ for $t > 100$. Correspondingly, in Fig.[2], we plot $a(t)$ and $T(t) \simeq 1/a(t)$, which show that (i) the pre-inflationary phase (14) for $t < 4$, (ii) an exponential inflation $a(t) \simeq a_0 \exp(N_e t)$ for $4 < t < 110$ and (iii) $a(t)$ approaching 10^{30} for $t > 110$.

We might consider that the inflation ends at $t \simeq t_h$, when the quantum black hole mass (11) is reduced to $M_H \simeq 1/(8\pi)$ corresponding to the black hole temperature $T_H \simeq 1$. We find that $t_h \simeq 113$, $a_h \simeq 2.84 \cdot 10^{28}$ and the e-folding factor $N_e t_h \simeq 65.5$.

The reheating process mainly occurs when $t \sim t_h$, where $|\Delta\rho_\Lambda(t)| \to \bar{\rho}_\Lambda$, $\rho(t) \to 0$ and scaling factor $a(t)$ slowly varies. The reheating temperature T_h can be possibly estimated by $g_s \frac{\pi^2}{15} T_h^4 \simeq |\Delta\rho_\Lambda(t_h)|$, where g_s stands for the summation over contributions of all relativistic particles created in the Hawking process. Since all possible relativistic particles are created, the reheating temperature $T_h < T_0 \simeq 1$. We leave T_h as a parameter in this letter. For $T_h \sim O(10^{-2})$, an enormous entropy $S_h = (T_h a_h)^3$ is produced.

These numerical values depend on the θ-value, the initial time t_0, temperature T_0 and scale factor a_0.

Cosmological constant. After reheating to the temperature T_h, the nucleation of quantum black holes starts again, and the density of energy-gain $\rho^h_\Lambda(t)$ is given by Eq.(7) with the initial time t_h and temperature T_h. Because the reheating temperature $T_h < T_0 \simeq 1$, $\rho^h_\Lambda(t)$ (7) is exponentially suppressed by lower temperature T_h. This possibly leads to $\rho^h_\Lambda \ll \rho_m \simeq T_h^4$ in Eq.(9). As a result, the Universe begins the evolution described by the Standard Cosmology:

$$a(t) = a_h(t/t_h)^\alpha, \quad T(t) = T_h(t/t_h)^{-\alpha}, \tag{15}$$

with total entropy $S_h = (a_h T_h)^3$. We consider that this is a new era initiated with t_h, T_h and a_h, independently from the inflationary era before the reheating.

The energy-density $\rho^h_\Lambda(t)$ is mainly contributed from quantum black holes nucleated in the reheating. Due to the Hawking radiation, the variation of energy-density $\Delta\rho^h_\Lambda(t)$ is given by Eq.(12) with initial time t_h and temperature T_h. Analogously to Eq.(13), the "dark-energy" density is,

$$\rho_h(t) = \rho^h_\Lambda(t) + \Delta\rho^h_\Lambda(t) < \rho^h_\Lambda(t), \tag{16}$$

which is related to the cosmological constant $\Lambda = 8\pi G\rho_h(t)$. Assuming that the mass of quantum black holes has been reduced to the minimal mass $M_H = 1/(8\pi)$ at the present time $t \gg t_h$, we obtain,

$$\rho_h(t) = \frac{0.87 m_p^4}{1024\pi^4}\int_{t_h}^t d\tau \left(\frac{1}{T(\tau)}\right)^{(\theta-1)} e^{-\frac{1}{16\pi T^2(\tau)}}, \tag{17}$$

from Eqs.(7,12) and (16). Substituting solutions $a(t)$ and $T(t)$ (15) into $\rho_h(t)$, we have,

$$\begin{aligned} \rho_h(t) &= \frac{0.87 m_p^4}{1024\pi^4}\int_{t_h}^t d\tau \left(\frac{\tau^\alpha}{T_h t_h^\alpha}\right)^{(\theta-1)} e^{-\frac{\tau^{2\alpha}}{16\pi T_h^2 t_h^{2\alpha}}} \\ &= \frac{0.87 m_p^4}{1024\pi^4}\frac{(16\pi)^\delta t_h}{2\alpha} T_h^{1/\alpha}\Gamma(\delta, z_2, z_1) \end{aligned} \tag{18}$$

where $\delta = [\alpha(\theta-1)+1]/(2\alpha)$, $z_2 = t^{2\alpha}/(16\pi T_h^2 t_h^{2\alpha})$ and $z_1 = 1/(16\pi T_h^2)$ in the incomplete Gamma-function,

$$\Gamma(\delta, z_2, z_1) \equiv \int_{z_1}^{z_2} dx x^{\delta-1} e^{-x} \simeq z_1^{\delta-1} e^{-z_1}. \tag{19}$$

The asymptotic representation of Eq.(19) is for $z_2 \gg z_1 \gg 1$ and $z_2 \to \infty$. Using this asymptotic representation, we approximately have a constant energy-density:

$$\rho_h \simeq \frac{0.87}{64\pi^3}\frac{m_p^4 t_h}{2\alpha} T_h^{(3-\theta)} e^{-\frac{1}{16\pi T_h^2}}, \tag{20}$$

whose numerical value crucially depends on the reheating temperature T_h.

We find that $\rho_h(t)$ and $\Lambda(t)$ increase, when the Universe is in radiation domination ($t > t_h, T < T_h$), and asymptotically approaches the constant (20), when the Universe is in matter domination ($t \gg t_h, T \ll T_h$). With $t_h \simeq 113$, $t \sim 10^{61}$, $\alpha = 1/2$ in the radiation dominant phase, and $\theta = 203/45$ for the particle content of the Standard Model, we obtain $\rho_h \simeq 6.7 \cdot 10^{-120} m_p^4$ by setting $T_h \simeq 8.45 \cdot 10^{-3}$. This is consistent with present observations.

Some remarks. In vacuum gravitation, the mass-energy of classical matter is zero ($T_{ab} = 0$), while the zero-point energy of quantum fields is non-zero ($T_{ab}^{\rm vec} \neq 0$). It seems that $T_{ab}^{\rm vec}$ possibly accounts for non-vanishing cosmological term in Einstein equation (8). However, the positive-energy theorem[5] actually requires zero mass-energy in the Minkowski spacetime. This implies that the zero-point energy of quantum-fields in the Minkowski spacetime is not gravitating and should not be related to the cosmological term in Einstein equation (8).

We consider that the cosmological term is originated from the vacuum-energy variation attributed to the unstable quantum fluctuation ($\epsilon\tilde{\phi}_{ab}$) about the classical gravitational field (g_{ab}^S), i.e., the decay of flat space at finite temperature proceeds by nucleating quantum black holes. Based on such a cosmological term, we study the evolution of the early Universe and present value of the cosmological constant.

The density perturbation that leads to the rich content of the present Universe is possibly originated from quantum fluctuations and Hawking radiation of quantum black holes nucleated in the inflationary phase, which will be presented in a future work.

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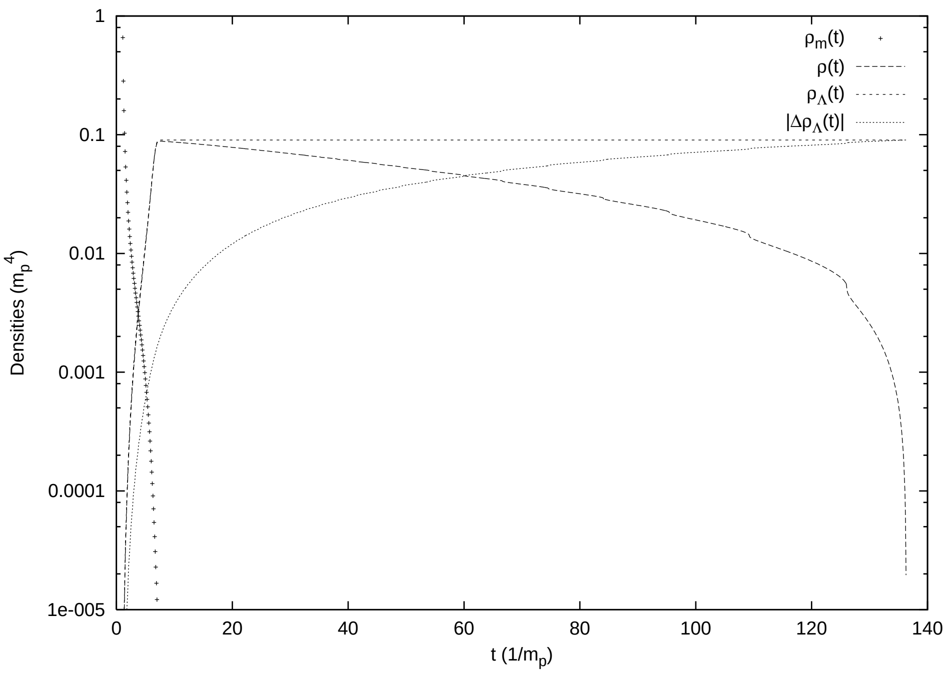

FIG. 1: The energy-densities $\rho_m(t)$, $\rho_\Lambda(t)$, $\rho(t)$ and $|\Delta\rho_\Lambda(t)|$ as functions of time.

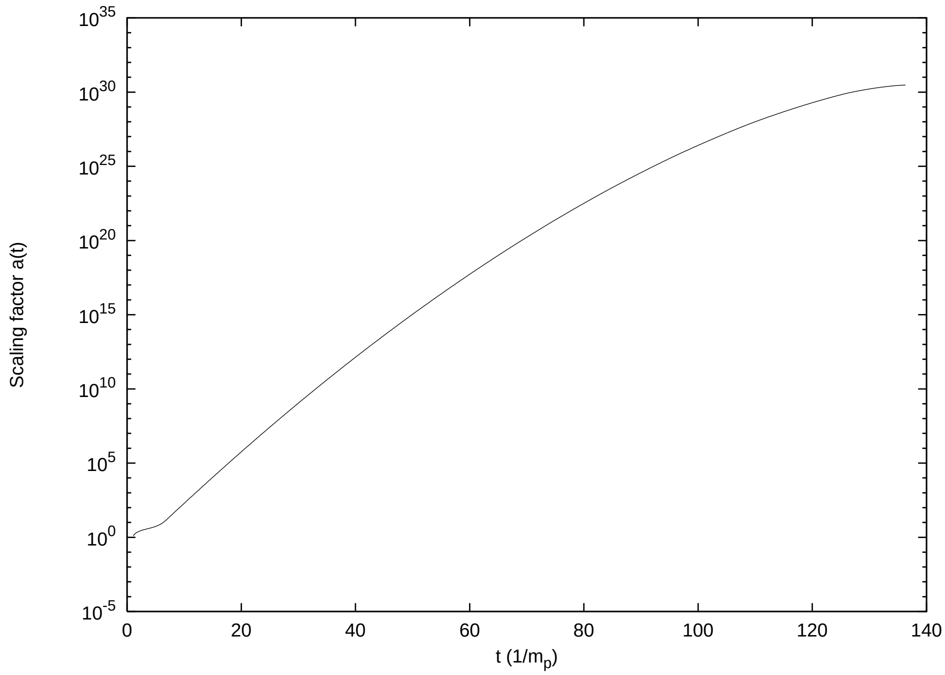

FIG. 2: The scaling factor $a(t)$ and temperature $T(t) = 1/a(t)$ as functions of time.